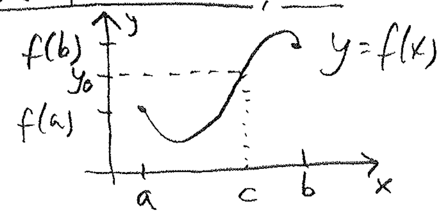
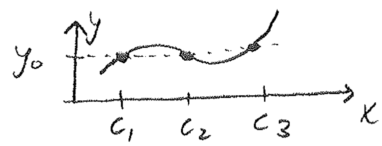
SHEET # 0651 : INTERMEDIATE AND MEAN VALUE THEOREMS (NOTES)

- INTERMEDIATE VALUE THEOREM FOR CONTINUOUS FUNCTIONS, IVT

A function $y=f(x)$ that is continuous on $[a,b]$ takes on every value between $f(a)$ and $f(b)$. If y_0 is between $f(a)$ and $f(b)$, then $y_0=f(c)$ for some c in $[a,b]$.



The value y_0 can be taken on for more than one x value.



EX. 1. Show that $f(x)=x^3-x-1$ has a zero between $x=1$ and $x=2$.

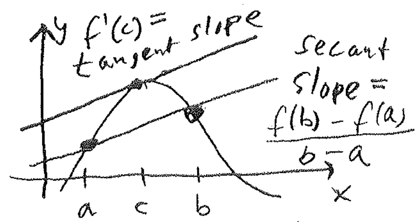
$f(1)=-1$, $f(2)=5$, $y_0=0$. By IVT there is a c such that $f(c)=0$.

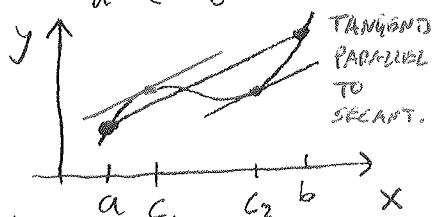
- MEAN VALUE THEOREM FOR DERIVATIVES, MVT

If $y=f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one point c in (a,b) at which the tangent slope =

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \text{secant slope},$$

or equivalently $f(b)-f(a) = f'(c)\cdot(b-a)$.





EX. 2. Show that $f(x)=x^2$ satisfies the hypotheses of the MVT on $[0,2]$. Then find c that solves $f'(c)=(f(b)-f(a))/(b-a)$.

$f(x)$ is continuous & differentiable. $f'(x)=2x$ So $f'(c)=2c=(f(2)-f(0))/(2-0)$

$2c=\frac{4-0}{2-0}$; $2c=2$. $\boxed{c=1.}$ So $f'(1)=2$ = Secant slope $=2$.

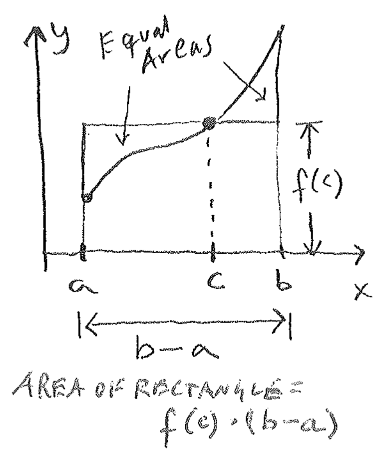
- MEAN VALUE THEOREM FOR DEFINITE INTEGRALS

If $y=f(x)$ is continuous on $[a,b]$ then at some point c

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = f_{ave}$$

or equivalently $\int_a^b f(x)\,dx = f(c)\cdot(b-a)$.

A continuous function always takes on its average value at least once on $[a,b]$.



Ex. 3. Find the average value of $f(x)=3x^2$ on $[0,2]$. At what point(s) on $[0,2]$ does $f(x)$ assume its average value?

$f_{ave} = \frac{1}{2-0}\int_0^2 3x^2\,dx = \frac{1}{2}\cdot x^3\Big|_0^2 = \frac{1}{2}\cdot 2^3 = \frac{8}{2}=4$. Solve $f(c)=4$.

$3c^2=4$. $c=\pm 2/\sqrt{3}$ Only $2/\sqrt{3}$ is in $[0,2]$, so $\boxed{c=2/\sqrt{3}}$. $f\left(\frac{2}{\sqrt{3}}\right)=4$.