Sheet #1240: Calculus Course Review

CALCULUS AB Review

RL 4/20/08
v.2

LIMITS

L'HÔPITAL'S RULE (B62)

$0/0$, ∞/∞, $0 \cdot \infty$ (NOT $\frac{1}{0}$)

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Ex.

$$\lim_{x\to 0} \frac{(\cos x - 1)}{x} = \lim_{x\to 0} \frac{\sin(x)}{1} = 0$$

RATIONAL FUNCTION THM (B28)

$$\lim_{x\to\infty} \frac{P(x)}{R(x)} = \begin{cases} 0 & \text{if } R \text{ has larger degree} \\ \frac{a_n}{b_n} & \text{if } P\&R \text{ have same degree} \\ \pm\infty & \text{if } P \text{ has larger degree} \end{cases}$$

Ex.

$$\lim_{x\to\infty} \frac{3x^2+1}{4x^3+2x^2} = 0$$

$$\lim_{x\to\infty} \frac{3x^2+1}{4x^2+2x} = \frac{3}{4}$$

$$\lim_{x\to\pm\infty} \frac{3x^3+1}{4x^2+2x^2} = \lim_{x\to\pm\infty} 3x \Rightarrow \pm\infty$$

CONTINUITY f CONTINUOUS AT $x=c$ IF $\lim_{x\to c} f(x) = f(c)$.

DIFFERENTIATION nDeriv(f, x, a). x means dx.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$h = \Delta x$

$h = x - a = \Delta x$
$x = a + h$

LIMIT MUST EXIST AT a OR FOR ALL x IN DOMAIN.

Ex.

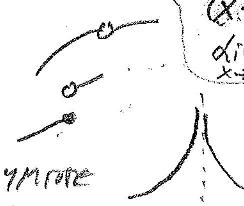
$$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)} = x+1, \quad x \neq 1$$

zoom 4 $\frac{2}{0}$

$\lim_{x\to 1} f(x) = 1+1 = 2$

NOT DIFFERENTIABLE:

1) Vertical tangent
2) Corner
3) Cusp
4) Not Continuous
a) hole
b) jump
c) Vertical Asymptote



to oscillating

ESTIMATES:

x	1.0	1.2	1.4	1.6
$f(x)$	8	10	14	22

Ex.

$$f'(1.5) \approx \frac{\Delta y}{\Delta x} = \frac{f(1.6)-f(1.4)}{1.6-1.4} = \frac{22-14}{0.2} = 5 \cdot 8 = \boxed{40}$$

Symmetric Diff quotient.

LINEARIZATION

POINT SLOPE FORMULA: $y - y_1 = m(x - x_1)$

$f(x) - f(a) = f'(a)(x-a)$ tangent line

APPROX.: $f(x) \approx f(a) + f'(a)(x-a)$

Ex.

If $f(x) = \frac{1}{1+x}$, $x \approx 0$

$$f(x) \approx 1 + \frac{-1}{(1+0)^2}(x-0) = 1 - x$$

$$f(0.1) = \frac{1}{1.1} \approx 1 - 0.1 = \boxed{0.9}$$

MEAN VALUE THM FOR DERIVATIVES

Secant line is parallel to tangent line for at least one c in (a,b)

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \text{AVERAGE RATE OF CHANGE OF } f \text{ over } [a,b]$$

if f continuous on $[a,b]$

Ex. $f(x) = x^2$ on $[0, 2]$

$$2c = \frac{2^2 - 0^2}{2-0} = \frac{4}{2} = 2. \quad c = 1$$

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RULES

TABLE (Blue Sheet)

DIFFERENTIATION

3 $\frac{d}{dx} X^N = N X^{N-1}$

13 $\frac{d}{dx} \ln(x) = \frac{1}{x}$

7 $\frac{d}{dx}(\sin x) = \cos x$

5 $\frac{d}{dx}(uv) = u'v + uv'$

6 $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$

Ex.

$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}$

$\int \frac{1}{\cos^2 x} dx \neq \ln(\cos^2 x) + c$
$= \tan x + c$

INDEFINITE INTEGRALS

ANTIDERIVATIVE + C.

I3 $\int x^N dx = \frac{x^{N+1}}{N+1} + C$

I4 $\int \frac{1}{x} dx = \ln|x| + C$

I6 $\int \sin x \, dx = -\cos x + C$

CHAIN RULE

B $\frac{d f(u)}{dx} = \frac{d f(u)}{du} \cdot \frac{du}{dx}$

$[f(g)]' = f'(g(x)) \cdot g'(x)$

Ex.

$\frac{d}{dx} e^{\cos x} =$
$= e^{\cos x} \cdot (-\sin x)$.

U-SUBSTITUTION

$\int f(u) du = \int f(u) \frac{du}{dx} dx$
$= \int f(u) \cdot u'(x) dx$

F *INVERSES*, If $u(x) = f^{-1}(x)$

$\frac{du}{dx} = \frac{1}{\frac{d}{du} f(u)}$

TABLE

$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=2} = \frac{1}{0.5} = 2$

x	$f(x)$	$f'(x)$
1	2	0.5
2	3	1

Ex.

$\int \cos(x^3) x^2 dx$

$= \int \cos(u) \cdot (3x^2) \frac{du}{3}$

$= \frac{1}{3}\int \cos(u) du = \frac{-1}{3} \sin x^3 + c$

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IMPLICIT DIFFERENTIATION

$$\frac{d}{dx} f(y) = \frac{d}{dy} f(y) \cdot \frac{dy}{dx}$$

Solve for dy/dx.

Ex.

CIRCLE: $x^2 + y^2 = R^2$

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

Hyperbola: $x^2 - y^2 = 1$ $\quad y' = \frac{x}{y}$

Curve: $x^2 + 2xy + y^3 = 9$

FIND y' AT $(2,1)$

$$2x + 2(y + xy') + 3y^2 y' = 0$$

$$4 + 2 + 4y' + 3y' = 0$$

$$7y' = -6 \quad y' = \frac{-6}{7}$$

RELATED RATES

- FIND EQUATION THAT RELATES QUANTITIES WITH RATES.
- TAKE DERIVATIVE WITH RESPECT TO TIME $d/dt\,(f(t))$
- USE CHAIN RULE.

Ex. $\frac{d}{dt}(y^2) = 2y \cdot \frac{dy}{dt}$

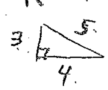
- USE PRODUCT RULE

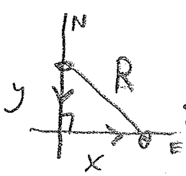
$$\frac{d}{dt}(R^2 \cdot H) = 2R\frac{dR}{dt} \cdot H + R^2 \cdot \frac{dH}{dt}$$

- DON'T FORGET THAT SOME RATES, SUCH AS VELOCITY, HAVE A DIRECTION: $\frac{ds}{dt} = v = \pm|v|$

Ex. Police approaching intersection from the north, speeding car moving east. When police is 0.6 mi north, car is 0.8 mi east, the dist between them is increasing at 20 mph, and the police speed is 60 mph. Find car speed.

$R = 1.0$ mi





$$R^2 = x^2 + y^2$$

$$2R\frac{dR}{dt} = 2x\frac{dx}{dt} - 2y\frac{dy}{dt}$$

$$R \cdot R' = x \cdot x' - y \cdot y'$$

$$1.0 \cdot 20 = 0.8x' - 0.6 \cdot 60$$

$$20 + 36 = 0.8x'$$

$$\boxed{x' = \frac{56}{0.8} = 70\text{ mph}}$$

COMMON FORMULAS: Pythagorean thm: $x^2 + y^2 = R^2$

Volume of sphere: $V = \frac{4}{3}\pi R^3$

Volume of cone $= \frac{1}{3}\pi R^2 \cdot H$. ← OFTEN USED WITH SIMILAR TRIANGLES.

CALCULUS REVIEW, CONTINUED

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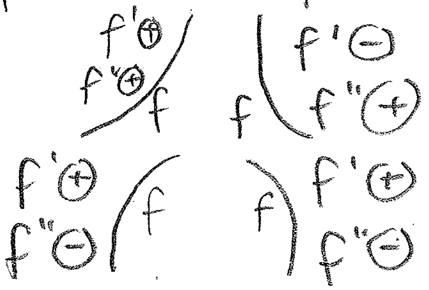
MAX/MIN f, f', f''

f INCREASING : $f' > 0$

f CONCAVE UP : $f'' > 0$, f' increasing

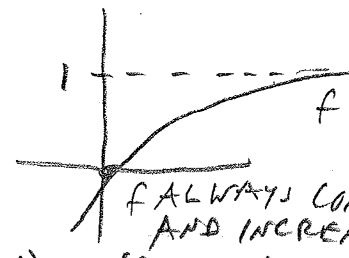


Keep in Calculator:

f	$y_1 = f(x)$
f'	$y_2 = \text{nDeriv}(y_1, x, x)$
f''	$y_3 = \text{nDeriv}(y_2, x, x)$

↑ POINT

Ex. $f(x) = 1 - e^{-x}$



f ALWAYS CONCAVE DOWN AND INCREASING.

Asymptote : $y = 1$

$\lim_{x \to \infty} f(x) = 1$. $\lim_{x \to -\infty} f(x) = -\infty$

MAX/MIN

ALL MAX/MIN ARE LOCAL = RELATIVE $[a, b]$

- INTERIOR : CRITICAL POINTS

  $f' = 0$, $f' =$ UNDEFINED

  NOT ALL CRIT. PTS ARE MAX/MIN (a, b)

- ENDPOINTS : $x = a$, $x = b$.

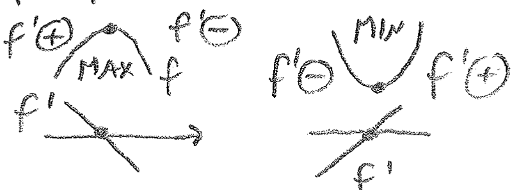
GLOBAL = ABSOLUTE MAX/MIN ARE AT CRIT. PTS AND/OR ENDPTS.

EXTREME VALUE THEOREM:

If f is continuous on $[a, b]$, f has both GLOBAL MAX & MIN.

FIRST DERIVATIVE TEST for f.

f' CHANGES SIGN :



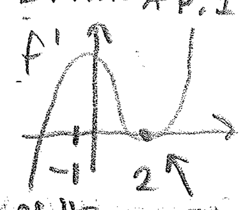
SECOND DERIVATIVE TEST :

If f'' exists, $f'' < 0$ CONCAVE DOWN $f'' > 0$ CONCAVE UP

Ex.

$f' = (x+1)(x-2)^2$

- f' SIGN CHANGE AT $X = -1$, SO f HAS A MIN.
- NO SIGN CHANGE FOR $f'(x)$ AT $X = 2$. f HAS NO MAX/MIN; IT HAS A P.I.



f' "BOUNCE"

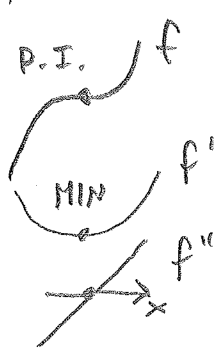
MANY QUESTIONS HAVE f' GIVEN, NOT f.

POINT OF INFLECTION, "P.I."

f HAS A CHANGE IN CONCAVITY

f' HAS A MAX OR MIN

f'' CHANGES SIGN



Ex.

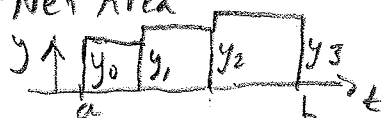
$y = (x+1)^3 - 1$

P.I. FOR $X = -1$.

R.S. 4/22/08

DEFINITE INTEGRALS

$I = \int_a^b f(t)\,dt$ = "Net Area"



ESTIMATES

- Left sum: $y_0 \cdot (\text{change in } t) + y_1(\text{change in } t) + y_2(\text{change in } t)$

  if Δt = constant $L_3 = (y_0 + y_1 + y_2)\Delta t$, $\Delta t = \frac{b-a}{3} \leftarrow n$

- Right sum: Don't use y_0, use y_3.
- TRAPEZOID: $\frac{\text{Left} + \text{Right}}{2} = \frac{1}{2}(y_0+y_1)\cdot(\text{change in } t) + \dots + \frac{1}{2}(y_2+y_3)(\text{change in } t)$

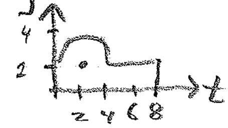
  if Δt = constant: $T_3 = \frac{1}{2}(y_0 + 2y_1 + 2y_2 + y_3)\cdot\Delta t$.

  T_n overestimates if f is concave up.

GEOMETRY

Use Shapes: $L\cdot W$, $\frac{1}{2}B\cdot H$, $\frac{1}{2}(b_1+b_2)\cdot H$, πr^2, ...

Ex.



$I = 2\cdot 8 + \frac{1}{2}\pi 2^2$.

FUNDAMENTAL THEOREM: EVALUATION

If $F(x)$ is any antiderivative, $\frac{dF(t)}{dt} = f(t)$

$I = \int_a^b f(t)\,dt = F(b) - F(a)$.

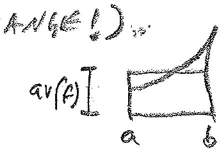
Ex. $\int_2^4 x^2\,dx = \left.\frac{x^3}{3}\right|_2^4 = \frac{4^3}{3} - \frac{2^3}{3} = 56/3$.

NUMERICAL ESTIMATES WITH CALCULATOR

$I = \text{fnInt}(f, x, a, b)$ x means dx. $\text{fnInt}(x^2, x, 2, 4) \cong 18.667$

AVERAGE VALUE (NOT RATE OF CHANGE!)

$av(f) = \frac{1}{b-a}\int_a^b f(t)\,dt$.



Areas Same. $\Delta x = b - a$

Ex. $av(x^2)$ on $[2,4]$ $= \frac{1}{4-2}\cdot\left(\frac{56}{3}\right) = \frac{56}{6}$

NET CHANGE

$F'(t) = f(t)$,

$\Delta F = \int_a^b f(t)\,dt = \int_a^b F'(t)\,dt$

Ex. If $F'(x) = x^2$ FROM 2 to 4: $\Delta F = \int_2^4 x^2\,dx = \frac{56}{6}$

THEOREMS

$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$

$\int_a^k f(t)\,dt + \int_k^b f(t)\,dt = \int_a^b f(t)\,dt$.

k does not have to be between a AND b.

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INTEGRAL FUNCTION

FTC: CONSTRUCTION/ANTIDERIVATIVE

$$F(x) = F(a) + \int_a^x f(t)\,dt$$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

DO NOT NEED ANALYTICAL FORMULA FOR $F(x)$.

Ex.

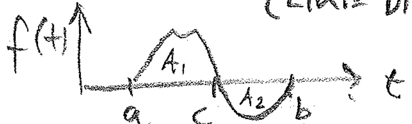
$$\frac{d}{dx}\int_1^{x^2} \cos t\,dt = \cos(x^2)\cdot 2x.$$

CHAIN RULE $\frac{d}{dx}F(u) = \frac{d}{dx}\int_a^u f(t)\,dt = f(u)\cdot\frac{du}{dx}.$

AREAS

$$A = \int_a^b |f(t)|\,dt > 0$$

(LIKE DISTANCE)



$$A = A_1 - A_2 = \int_a^c f(t)\,dt - \int_c^b f(t)\,dt$$

AREA BETWEEN CURVES.

$$A = \int_a^b f(t) - g(t)\,dt$$

$f \geq g.$

MOTION

s = position, $v = \frac{ds}{dt}$ = Velocity, $a = \frac{d^2s}{dt^2} = \frac{dv}{dt}$ Acceleration

Δs = change in position (= displacement)

$$= \int_a^b v(t)\,dt \quad \text{(LIKE INTEGRAL)}$$

$$s(t) = s(a) + \int_a^t v(x)\,dx$$

$$D = \text{distance} = \int_a^t |v(t)|\,dt \quad \text{(LIKE AREA)}$$

$|v|$ = speed.

CONSTANT ACCELERATION

$$s = \tfrac{1}{2}at^2 + v_0 t + s_0 \qquad v_0 = v(t=0),\ s_0 = s(t=0)$$

Ex.

Particle $v(t) = t^2$. Find $s(t)$ and $s(4)$ if position is $\frac{4}{3}$ at $t=2$.

$$s(t) = s(2) + \int_2^t x^2\,dx$$

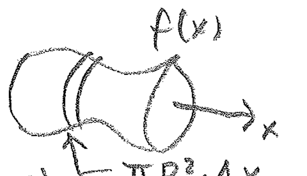
NET CHANGE: see page R5.

$$s(4) = s(2) + \int_2^4 x^2\,dx = \underbrace{\frac{4}{3}}_{\text{GIVEN POSITION}} + \frac{56}{3} = \frac{60}{3} = 20$$

VOLUMES

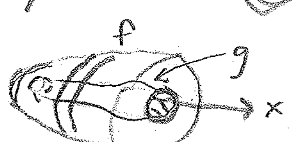
Solid of Revolution

DISKS $V = \int_a^b \pi\left(f(x)\right)^2 dx$



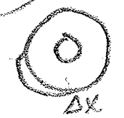
WASHERS $V = \int_a^b \pi\left(f(x)^2 - g(x)^2\right) dx$

KNOWN AREAS ON A BASE.



$$V = \int_a^b A(x)\,dx$$

$R = f(x)$
$r = g(x)$



SQUARES: $V = \int_a^b \left(f(x) - g(x)\right)^2 dx.$

Ex. Solid of Revolution (x-axis)

$f(x) = \cos x,\ g(x) = \sin x.$

INTERSECTION: $x = \pi/4$

$$V = \int_0^{\pi/4} \pi\left(\cos^2 x - \sin^2 x\right) dx$$

$= \dots = \pi/2.$

Ex. Semicircles Stacked on base $y=0$, $y = \sin(x)$.

DIAMETER: $\sin(x)$

R = RADIUS $= \frac{1}{2}\sin(x)$

$V_{SLICE} = \frac{1}{2}\pi R^2 \cdot \Delta x$

$$V = \int_0^{\pi} \tfrac{1}{2}\pi\left(\frac{\sin x}{2}\right)^2 dx$$

$\dots = \pi^2/16 \approx 0.617.$

R7. 4/22/08

U. 2.

DIFFERENTIAL EQUATIONS

$dy/dx = f(x)$, $y(a) = f(a) = d$

GENERAL SOLUTION:

$y(x) = \int f(x)\,dx = F(x) + C$

PARTICULAR SOLUTION: Solve for C

USING FTC

$$y(x) = \underbrace{y(a)}_{\text{"INITIAL" CONDITION}} + \underbrace{\int_a^x f(t)\,dt}_{\text{NET CHANGE}}$$

Ex.

Solve $\frac{dy}{dx} = 4x^3$, $y(1) = 5$

- $y(x) = \int 4x^3\,dx = x^4 + c$

  $5 = 1^4 + c \rightarrow y = x^4 + 4$

- $y(x) = 5 + \int_1^x 4t^3\,dt$

  $= 5 + (x^4 - 1^4) = x^4 + 4$

Ex.

Solve $\frac{dy}{dx} = e^{x^2 + \sin x}$, $y(0) = 2$

$y(x) = 2 + \int_0^x e^{t^2 + \sin t}\,dt$

$\rightarrow$ use calculator!

SEPARABLE DIFF. EQ'S.

Rate proportional to amount y

$dy/dt = ky$

$y = Ae^{kt}$ A = initial AMOUNT

k = CONTINUOUS Rate constant

$\frac{dy}{dx} = g(x) \cdot h(y)$

$\int \frac{dy}{h(y)} = \int \frac{dx}{g(x)}$

Solve for $y(x)$.

Ex. $\frac{dy}{dx} = xy$

$\int \frac{dy}{y} = \int x\,dx$

$\ln|y| = \frac{x^2}{2} + C_1$

$y = e^{x^2/2 + C_1}$

$y = Ae^{x^2/2}$, $A = e^{C_1}$

CHECK: $\frac{dy}{dx} = (Ae^{x^2/2})\left(\frac{2x}{2}\right)$

$= y \cdot x$

SLOPE FIELDS

IMAGINE 2-SIDED FLAGS

BLOWING IN DIRECTION OF WIND WITH

DIRECTION (SLOPE) = $\frac{dy}{dx} = f(x,y)$.

A PARTICULAR SOLUTION IS

UNIQUE AND FOLLOWS THE WIND

LIKE A BALLOON.

SLOPE FIELD:

x	y	$dy/dx = xy$
0	ANY	0
ANY	0	0
1	1	1
2	1	2
1	2	2
−1	2	−2