SHEET #300:

CALCULUS DERIVATIVES

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FORMULAS TO KNOW

1. $\frac{d}{dx} x^n = \boxed{n x^{n-1}}$

2. a, $\frac{d}{dx}(fg) = \boxed{fg' + gf'}$ $\qquad \frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

   b, $\frac{d}{dx}\left(\frac{f}{g}\right) = \boxed{\frac{gf' - fg'}{g^2}}$ $\qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ [DERIVED IN A.]

3. $\frac{d}{dx} e^x = \boxed{e^x}$

4. $\frac{d}{dx} \ln(x) = \boxed{\frac{1}{x}}$ [DERIVED IN N.]

5. $\frac{d}{dx} \sin(x) = \boxed{\cos(x)}$

6. $\frac{d}{dx} \cos(x) = \boxed{-\sin(x)}$

7. $\cos^2 x + \sin^2 x = 1$

8. [CHAIN RULE]
   IF $y = f(u)$ AND $u = g(x)$

   a, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
   evaluated at $u = g(x)$.

   b, $\boxed{\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).}$

   c,* $\frac{d}{dx} f(u) = \frac{df(u)}{du} \cdot \frac{du}{dx}$

   d, $\frac{d}{dx} f(u) = \left.\frac{d\,f(x)}{dx}\right|_u \cdot \left.\frac{du}{dx}\right|_x$

9. [IMPLICIT DIFFERENTIATION]
   - DIFFERENTIATE WITH RESPECT TO x
   - USE CHAIN RULE: $\frac{d}{dx} f(y) \overset{8c}{=} \frac{d\,f(y)}{dy} \cdot \frac{dy}{dx}$
   - SOLVE FOR dy/dx

10. [INVERSES]

$$\boxed{\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{\frac{d}{dx}\left[f(f^{-1}(x))\right]}}$$

WHERE $(x, f^{-1}(x))$ IS A COORDINATE PAIR

a, AT THE POINT $(f(a), a)$

$\xrightarrow{8d}$ $\left.\frac{d}{dx} f^{-1}(x)\right|_{f(a)} = \frac{1}{\left.\frac{df(x)}{dx}\right|_a}$

b,* AT THE POINT $(b, f^{-1}(b))$

$\xrightarrow{8d}$ $\left.\frac{d}{dx} f^{-1}(x)\right|_b = \frac{1}{\left.\frac{d}{dx} f(x)\right|_{f^{-1}(b)}}$

c,* IF $u = f^{-1}(x)$

$$\frac{du}{dx} = \frac{1}{\frac{d\,f(u)}{du}}$$

[DERIVED IN O.]

*USEFUL IN PRACTICE

FORMULAS TO DERIVE

A. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}(u \cdot v^{-1}) \overset{2a}{=} \frac{du}{dx}\cdot v^{-1} + u \cdot \frac{d}{dx}(v^{-1}) \overset{1,8}{=} \frac{du}{dx}\cdot v^{-1} - u\cdot v^{-2}\cdot\frac{dv}{dx} =$ ← Use formula 2a

$= \dfrac{\frac{du}{dx}\cdot v - u\frac{dv}{dx}}{v^2}$ [FORMULA 2b]

B. $\frac{d}{dx} a^x = \frac{d}{dx}\left[e^{\ln(a^x)}\right] = \frac{d}{dx} e^{x\ln(a)} \overset{3,8}{=} \ln(a)\, e^{x\ln(a)} = \boxed{\ln(a)\cdot a^x}$ ← TRICK: $a^x = e^{\ln(a^x)} = e^{x\ln(a)}$

C. $\frac{d}{dx}\log_a(x) = \frac{d}{dx}\left[\frac{\ln(x)}{\ln(a)}\right] \overset{4}{=} \boxed{\frac{1}{x\cdot\ln(a)}}$

D. $\frac{d}{dx}\tan(x) = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] \overset{2b}{=} \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} \overset{7}{=} \frac{1}{\cos^2(x)} = \boxed{\sec^2(x)}$

E. $\frac{d}{dx}\cot(x) = \boxed{-\csc^2(x)}$ [SEE FORMULA D.]

F. $\frac{d}{dx}\sec(x) = \frac{d}{dx}\left[\cos(x)\right]^{-1} \overset{1,8}{=} (-1)\left[\cos(x)\right]^{-2}\cdot(-\sin(x)) = \frac{\sin x}{\cos^2 x} = \boxed{\sec(x)\cdot\tan(x)}$

G. $\frac{d}{dx}\csc(x) = \boxed{-\csc(x)\cdot\cot(x)}$ [SEE FORMULA F.]

H. $\frac{d}{dx}\left[\sin^{-1}(x)\right]$. LET $f(x) = \sin(x)$ AND $u = f^{-1}(x) = \sin^{-1}(x)$.

NOTE: $f(u) = \sin(u)$ AND $f(f^{-1}(x)) = x = f(u)$.

$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{du}{dx} \overset{10c}{=} \frac{1}{df(u)/du} \overset{5}{=} \frac{1}{\cos(u)} \overset{7}{=} \frac{1}{\sqrt{1-\sin^2 u}} = \boxed{\frac{1}{\sqrt{1-x^2}}},\ |x|<1.$

I. $d/dx\left[\cos^{-1}(x)\right] = \boxed{-\,d/dx\left[\sin^{-1}(x)\right]}$ [SEE FORMULA H. OR USE $\cos^{-1}(x) = \pi/2 - \sin^{-1}(x)$.]

J. $\frac{d}{dx}\left[\tan^{-1}(x)\right] \overset{10c,\,D}{=} \frac{1}{\sec^2(u)} \overset{7}{=} \frac{1}{1+\tan^2(u)} = \boxed{\frac{1}{1+x^2}}$ [USING $f(u) = \tan u$]

K. $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \boxed{-\,d/dx\left[\tan^{-1}(x)\right]}$ [SEE FORMULA J. OR USE $\cot^{-1}(x) = \pi/2 - \tan^{-1}(x)$].

L. $\frac{d}{dx}\left[\sec^{-1}(x)\right] \overset{10c,\,F}{=} \frac{1}{\sec(u)\cdot\tan(u)} \overset{7}{=} \frac{1}{\sec(u)\cdot\sqrt{\sec^2(u)-1}} = \boxed{\frac{1}{|x|\sqrt{x^2-1}}},\ |x|>1$

↑ SLOPE IS ALWAYS POSITIVE

M. $\frac{d}{dx}\left[\csc^{-1}(x)\right] = \boxed{-\,d/dx\left[\sec^{-1}(x)\right]}$ [SEE FORMULA L. OR USE $\csc^{-1}(x) = \pi/2 - \sec^{-1}(x)$.]

N. $\frac{d}{dx}\left[\ln(x)\right]$. LET $f(x) = e^x$ AND $u = f^{-1}(x) = \ln(x)$.

NOTE: $f(u) = e^u$ AND $f(f^{-1}(x)) = x = f(u)$.

$\frac{d}{dx}\left[\ln(x)\right] = \frac{du}{dx} \overset{10c}{=} \frac{1}{df(u)/du} = \frac{1}{e^u} = \frac{1}{x}$ [FORMULA 4.]

O. $\frac{d}{dx}\left[f^{-1}(x)\right]$. LET $u = f^{-1}(x)$. FIND du/dx. NOTE: $f(u) = x$ SINCE $f(f^{-1}(x)) = x$.

$f(u) = x$

$\overset{1}{\Downarrow}\ \frac{df(u)}{dx} = 1$

$\overset{8c}{\Downarrow}\ \frac{df(u)}{du}\cdot\frac{du}{dx} = 1$

$\frac{du}{dx} = \frac{1}{df(u)/du}.$

[FORMULA 10c]

THIS IS THE SAME AS

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{\frac{d}{dx}\left[f(f^{-1}(x))\right]} = \frac{1}{\left.\frac{d}{dx}f(x)\right|_{f^{-1}(x)}}$$

FOR THE COORDINATE PAIR $(x, f^{-1}(x))$.

[FORMULA 10.]